Sobolev estimates for the complex Green operator on Levi-flat manifolds

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Abstract. This note is an announcement of the author's recent work on analysis of the complex Green operator for CR line bundles over compact Levi-flat manifolds. We explain a special but interesting example, the unit tangent bundle of hyperbolic Riemann surface with leafwise pluricanonical bundle, where Sobolev estimates for complex Green operator hold up to certain order, but fail to the order above.

1 Introduction

The aim of this study is to develop function theory on Levi-flat manifolds:

Definition 1.1. An odd-dimensional smooth manifold M is said to be *Levi-flat* if it admits a real codimension one smooth foliation $\mathcal{F} = \{L_\lambda\}_{\lambda\in\Lambda}$, called the *Levi foliation* of M, endowed with leafwise complex structure, namely, a smooth bundle map $J\colon T\mathcal{F} \to T\mathcal{F}$ such that $J|TL_\lambda$ gives an integrable complex structure of the leaf L_λ for all $\lambda \in \Lambda$.

Definition 1.2. A measurable function $f\colon M \to \mathbb{R}$ is said to be *CR* if its restriction on any leaf L_λ is holomorphic.

The simplest example of Levi-flat manifolds is the direct product $X \times \mathbb{R}$ of a complex manifold X with the real line $\mathbb{R}$. In this case, function theory on this Levi-flat manifold is just that on the complex manifold X with real parameter. When the Levi foliation $\mathcal{F}$ has non-trivial dynamics, however, we will see subtle interaction between function theory on the leaves of $\mathcal{F}$ and transverse dynamics of $\mathcal{F}$.

The starting point of this study is the following theorem.

Theorem 1.3 (Inaba [6])**.** *Let M be a compact Levi-flat manifold. Then, any CR function which is continuous on M must be constant on all the leaves. In particular, when the Levi foliation has a dense leaf, any continuous CR function must be constant on M.*

Hence, when M is compact and the Levi foliation $\mathcal{F}$ has a dense leaf, function theory on M is expected to be similar to that on compact complex manifolds. Therefore, it would be natural to study "CR meromorphic functions", which is formulated as CR sections of CR line bundle in this note (see [3] for the subtlety appearing in defining CR meromorphic functions).

Definition 1.4. A smooth $\mathbb{C}$-line bundle B on M is said to be *CR line bundle* if it is covered by a system of smooth local trivializations whose transition functions are all smooth CR. A section of B is said to be a *CR section* if it is CR in the local trivializations.

Then, we have the following analogue of Kodaira's embedding theorem.

Theorem 1.5 (Ohsawa–Sibony [9], see also Ohsawa [8], Hsiao–Marinescu [5])**.** *Let M be a compact Levi-flat manifold, and B a CR line bundle. Suppose that B is positive, namely, B admits a smooth hermitian metric whose curvature form on any leaf is positive definite everywhere on the leaf. Then, for any given $k \in \mathbb{N}$, there is $n = n(k) \in \mathbb{N}$ such that for any $m \geq n$,*

- *The space of C^k-smooth CR sections of $B^{\otimes m}$ is infinite dimensional;*
- *The bundle $B^{\otimes m}$ is C^k-very ample, namely, we may pick C^k-smooth CR sections of $B^{\otimes m}$, say, $s_0, s_1, \dots, s_N$, so that the ratio $[s_0 : s_1 : \cdots : s_N] : M \to \mathbb{CP}^N$ is an embedding.*

This follows from a Sobolev esimate of complex Green operator, which we shall explain in §3. An interesting point of this theorem is that the tensor power order $n(k)$ required to get a plenty of C^k-smooth CR sections of $B^{\otimes m}$, $m \geq n(k)$, does depend on the regularity k, as a previous study of the author revealed.

Theorem 1.6 ([1])**.** *There is an example of a compact Levi-flat manifold M, and a CR line bundle B over M for which the following statement holds: for any $n \in \mathbb{N}$, there exists $k = k(n) \in \mathbb{N}$ such that*

- *Any C^k-smooth CR section of $B^{\otimes n}$ automatically becomes C^∞-smooth;*
- *The space of C^k-smooth CR sections of $B^{\otimes n}$ is finite dimensional;*
- *We cannot make an embedding $[s_0 : s_1 : \cdots : s_N] : M \to \mathbb{CP}^N$ by picking C^k-smooth CR section $s_0, s_1, \dots, s_N$ of $B^{\otimes n}$.*

Especially, we cannot make C^∞-smooth CR projective embedding of M by using sections of the powers of B in this example.

Roughly speaking, Theorem 1.5 says that if the bundle $B^{\otimes n}$ has enough positive curvature and if we only look at CR sections of low transverse regularity, we always have a plenty of (actually, infinite dimensionally many) sections regardless of the dynamics of $\mathcal{F}$. Theorem 1.6 says that if we require CR sections to have better and better transverse regularity, then, a "phase transition" happens on some threshold line that reflects the dynamics of $\mathcal{F}$, beyond which the space of our CR sections shrinks to a finite dimensional vector space and shows rigid behavior. Figure 1 is the "phase diagram" summarizing this phenomenon.

The goal of this note is to explain this "phase transition" in detail for a special but interesting example, the unit tangent bundle of hyperbolic Riemann surface equipped with with leafwise pluricanonical bundle, which we explain in §2. In §3, we discuss its relation with Sobolev estimates of complex Green operator, from which Theorem 1.5 follows. In §4, we explain the threshold line of this example (Theorem 2.6) by describing L^2 CR sections in certain Fourier series, which is an application of the techniques developed in the author's recent work [2].

Remark 1.7. Typical non-trivial examples of Levi-flat manifolds arise as Levi-flat real hypersurfaces in complex manifolds, in particular, invariant real hypersurfaces of (singular) holomorphic foliations, which is the reason why Levi-flat manifolds have attracted

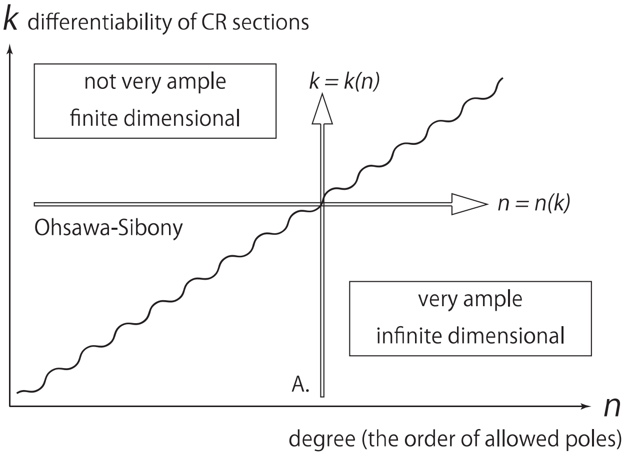

Figure 1: The space of C^k-smooth CR sections of $B^{\otimes n}$ in the example of [1]

people working in several complex variables and/or topology of foliations. In particular, it has been conjectured since 1980s that there is no smooth compact Levi-flat real hypersurface in complex projective plane $\mathbb{CP}^2$, which is an important part of *exceptional minimal set conjecture*: any leaf of a singular holomorphic foliation $\mathcal{F}$ of $\mathbb{CP}^2$ must have an accumulation point in the singular locus of $\mathcal{F}$. Note that exceptional minimal set conjecture is a complex counterpart of celebrated Poincaré–Bendixson's theorem. Several papers had claimed to solve the non-existence problem of Levi-flat real hypersurface in $\mathbb{CP}^2$, but up to 2018, no complete proof has been given. Interested readers are referred to [7] for the status of papers published before 2008.

2 Unit tangent bundle of Riemann surface

In this section, we shall give an explicit example to which both Theorems 1.5 and 1.6 are applied.

Let Σ be a compact Riemann surface of genus > 1. We endow Σ with the hyperbolic metric denoted by g. By Koebe–Poincaré uniformization theorem, the universal covering of Σ is isomorphic to the unit disk $\mathbb{D}$ and we may regard Σ as the quotient space of $\mathbb{D}$ by a Fuchsian group Γ, which is isomorphic to the fundamental group $\pi_1(\Sigma)$. Since each element of Γ is a linear fractional transformation, Γ also acts on the unit circle S^1, hence, on the direct product $\mathbb{D} \times S^1$ diagonally. Since this action of Γ on $\mathbb{D} \times S^1$ preserves the foliation $\{\mathbb{D} \times \{t\}\}_{t \in S^1}$, the quotient space $M := \mathbb{D} \times S^1/\Gamma$ becomes a compact Levi-flat 3-fold.

By the first projection, we have a map $\pi\colon M \to \Sigma = \mathbb{D}/\Gamma$, which gives the structure of circle bundle to M. This is actually the unit tangent bundle of Σ.

Proposition 2.1. *The Levi-flat 3-fold M is isomorphic to the unit tangent bundle $ST\Sigma$ of hyperbolic surface Σ as a circle bundle over Σ.*

Proof. For each unit tangent vector $v_p \in T_p\Sigma$, $p \in \Sigma$, consider the complete geodesic $\gamma_{v_p} \colon \mathbb{R} \to \Sigma$ with initial velocity $\gamma'_{v_p}(0) = v_p$. We take an arbitrary lift of γ_{v_p} to the universal covering $\mathbb{D}$ and obtain $\widetilde{\gamma_{v_p}} \colon \mathbb{R} \to \mathbb{D}$. The image $\widetilde{\gamma_{v_p}}(\mathbb{R})$ is an arc or segment that is perpendicular to the unit circle S^1, hence, $\lim_{t\to+\infty} \widetilde{\gamma_{v_p}}(t) \in S^1$ exists. Using this, we obtain a bundle map

$$ST\Sigma \to M = \mathbb{D} \times S^1/\Gamma, \quad v_p \mapsto [(\widetilde{\gamma_{v_p}}(0), \lim_{t\to+\infty} \widetilde{\gamma_{v_p}}(t))]$$

and we can easily see that this map is an isomorphism. □

Remark 2.2. The Levi foliation of M agrees with the unstable foliation of the geodesic flow of Σ.

Applying Theorem 1.3 to M, we know that any continuous CR function M is constant since the Levi foliation of M has a dense leaf. We can say more from a classical theorem on the dynamics of the geodesic flow of Σ, Hopf's ergodicity theorem.

Theorem 2.3 (Hopf [4]). *The geodesic flow of Σ is ergodic with respect to the Liouville measure. In other words, the diagonal action of Γ on $S^1 \times S^1$ is ergodic with respect to its Lebegues measure.*

Proposition 2.4. *Any bounded CR function on M is constant almost everywhere.*

Proof. Let f be a bounded CR function on $M = \mathbb{D} \times S^1/\Gamma$. This yields a bounded CR function F on $\mathbb{D} \times S^1$ that is invariant under the action of Γ. Since for each $t \in S^1$, $F(\cdot, t)$ is a bounded holomorphic function on $\mathbb{D}$, Fatou's theorem gives its boundary value function $\dot{F}$ on $S^1 \times S^1$ which is a bounded measurable function almost everywhere defined. From Theorem 2.3, this boundary value function $\dot{F}$ must be constant almost everywhere, and it follows that F is constant on almost all leaves and so is f. □

Now consider a CR line bundle over Σ given by the pull-back bundle $B := \pi^* K_\Sigma$, where K_Σ denotes the canonical line bundle, i.e., the cotangent bundle of Σ. The dual metric of the hyperbolic metric g gives a hermitian metric g^{-1} of K_Σ and it is positively curved on Σ. This metric g^{-1} induces a hermitian metric of B, still denoted by g^{-1}, and since $B = \pi^* K_\Sigma$ is a pull-back bundle, the metric g^{-1} is positively curved on any leaf of the Levi foliation of M. We are going to explain that this $B \to M$ enjoys not only Theorem 1.5 but also Theorem 1.6 in the following sense.

Definition 2.5. A CR section s of $B^{\otimes n}$ is said to be H^k, $k \in \mathbb{N}$, if for any $p \in \Sigma$, its restriction $s|M_p$ on the fiber $M_p := \pi^{-1}(p)$ belongs to $H^k(S^1)$, the L^2-Sobolev space of order k. Here we used a trivialization $B^{\otimes n}|M_p \simeq M_p \times S^1$ for $B^{\otimes n} = \pi^* K_\Sigma^{\otimes n}$ is a pull-back bundle.

Theorem 2.6. *The space of H^k CR sections of $B^{\otimes n}$ is infinite dimensional if and only if $n \geq k + 1$.*

The proof for Theorem 2.6 will be sketched in §4. Before that, we would like to explain its relation with Sobolev estimates of complex Green operator on M in the next section.

3 Complex Green Operator

In this section, we discuss the relation between Sobolev estimates of complex Green operator and the existence of CR sections of prescribed transverse regularity.

We usually use the Cauchy–Riemann equation, also called the $\overline{\partial}$-equation, on complex manifolds to produce holomorphic functions or sections with prescribed properties. On Levi-flat manifolds, we use the tangential Cauchy–Riemann equation, which is just the leafwise $\overline{\partial}$-equation, to produce CR sections. We restrict ourselves to explain this general procedure in our special case.

Let M and $B \to M$ be the example explained in §2. We denote by (z,t) the coordinate of the covering space $\mathbb{D} \times S^1$ of $M = \mathbb{D} \times S^1/\Gamma$, and identify sections of $B^{\otimes n}$ with Γ-invariant differential forms on $\mathbb{D} \times S^1$ in the form of

$$u = u(z,t)(dz)^{\otimes n}.$$

We denote the space of smooth section of $B^{\otimes n}$ by $\Omega^{0,0}(M, B^{\otimes n})$. Similarly, leafwise $(0,1)$-forms valued in $B^{\otimes n}$ are Γ-invariant differential forms on $\mathbb{D} \times S^1$ in the form of

$$v = v(z,t)(dz)^{\otimes n} \otimes d\overline{z},$$

and we denote the space of smooth leafwise $(0,1)$-forms valued in $B^{\otimes n}$ by $\Omega^{0,1}(M, B^{\otimes n})$. The tangential Cauchy–Riemann operator $\overline{\partial}_M : \Omega^{0,0}(M, B^{\otimes n}) \to \Omega^{0,1}(M, B^{\otimes n})$ is given by

$$\overline{\partial}_M u = \frac{\partial u}{\partial \overline{z}}(z,t)(dz)^{\otimes n} \otimes d\overline{z}.$$

Note that $\overline{\partial}_M u = 0$ is equivalent to say that u is CR.

We use the hyperbolic metric

$$g = g(z) i dz \wedge d\overline{z} := \frac{2i dz \wedge d\overline{z}}{(1-|z|^2)^2}$$

on $\Sigma = \mathbb{D}/\Gamma$, and a transverse measure of the Levi foliation given by

$$\mu = \mu(z,t)\frac{dt}{it} := \frac{1-|z|^2}{|t-z|^2}\frac{dt}{it} \left(= \frac{1-|z|^2}{|e^{i\theta}-z|^2} d\theta, \quad t = e^{i\theta} \right)$$

to metrize these two spaces of $B^{\otimes n}$-valued forms. We define the inner products

$$\langle\langle u_1, u_2 \rangle\rangle_{0,n} := \int_M \langle u_1, u_2 \rangle_g dV := \int_M u_1(z,t)\overline{u_2(z,t)} g(z)^{-n} dV$$

$$\langle\langle v_1, v_2 \rangle\rangle_{0,n} := \int_M \langle v_1, v_2 \rangle_g dV := \int_M v_1(z,t)\overline{v_2(z,t)} g(z)^{-(n+1)} dV$$

for $u_j \in \Omega^{0,0}(M, B^{\otimes n})$ and $v_j \in \Omega^{0,1}(M, B^{\otimes n})$ where $dV := g \wedge \mu$.

The completion of $\Omega^{p,q}(M, B^{\otimes n})$ with respect to the corresponding L^2-norm $\|\cdot\|_{0,n}$ is denoted by $L^{p,q}(M, B^{\otimes n})$. We consider the maximal closed extension of $\overline{\partial}_M$ on $L^{0,0}(M, B^{\otimes n})$, still denoted by $\overline{\partial}_M$, and denote its Hilbert adjoint by $\overline{\partial}_M^*$.

Ohsawa and Sibony [9] established the Nakano identity on Levi-flat manifolds and obtained the L^2-vanishing theorem of Akizuki–Nakano type. In our case, their theorem guarantees that the tangential Cauchy–Riemann equation is always L^2-solvable.

Proposition 3.1 (cf. [9, Proposition 1])**.** *For any $n \in \mathbb{N}$ and $v \in L^{0,1}(M, B^{\otimes n})$, there is a solution $u \in L^{0,0}(M, B^{\otimes n})$ to $\overline{\partial}_M u = v$.*

Proof. The Nakano identity of Ohsawa and Sibony yields a priori estimate

$$\|\overline{\partial}_M^* v\|_{0,n}^2 \geq \int_M \frac{1}{g} \frac{\partial^2(-\log g^{-(n-1)}(z)\mu(z,t))}{\partial z \partial \overline{z}} |v|_g^2 dV = (n-1+\frac{1}{2})\|v\|_{0,n}^2 \gtrsim \|v\|_{0,n}^2$$

for $v \in \Omega^{0,1}(M, B^{\otimes n})$. By the standard argument, this implies the desired L^2-solvability. □

Corollary 3.2. *For any $n \geq 2$, the space of L^2 CR sections of $B^{\otimes n}$ is infinite dimensional.*

Proof. Take a non-trivial holomorphic 1-form $s_0 \in H^0(\Sigma, K_\Sigma)$ and regard $s_0 (= s_0 \circ \pi$, more precisely) as a smooth CR section of $B = \pi^* K_\Sigma$. Pick a zero $p \in \Sigma$ of s_0 and we consider arbitrary smooth function $s(t)$ defined on the fiber $M_p \simeq S^1$. We shall find a L^2 CR section $\widetilde{s}$ of $B^{\otimes n}$ such that $\widetilde{s}(p,t)(dz)^{\otimes n} = s(t)(dz)^{\otimes n}$.

To do this, we first make a smooth extension $u \in \Omega^{0,0}(M, B^{\otimes n})$ by

$$u = u(z,t)(dz)^{\otimes n} := \chi(z)s(t)(dz)^{\otimes n}$$

on $R \times S^1$ where R is a fundamental region of Σ in $\mathbb{D}$ and $\chi \colon \mathbb{D} \to [0,1]$ is a smooth function satisfying

$$\chi(z) := \begin{cases} 1 & (0 \leq |z-p| \leq \varepsilon) \\ 0 & (|z-p| \geq 2\varepsilon) \end{cases}$$

for $0 < \varepsilon \ll 1$. We extend u on $\mathbb{D} \times S^1$ so that it is Γ-invariant. Although this u itself is not CR, we may modify this u to be CR. Since $\overline{\partial}_M u$ has support in $\{\varepsilon < |z-p| < 2\varepsilon\}$, where s_0 has no zero, $\overline{\partial}_M u/s_0$ is a well-defined smooth $(0,1)$-form on M valued in $B^{\otimes(n-1)}$. From Proposition 3.1, we have $u_1 \in L^2(M, B^{\otimes(n-1)})$ such that $\overline{\partial}_M u_1 = \overline{\partial}_M u/s_0$, that is, $\overline{\partial}_M(u - u_1 s_0) = 0$. Therefore, we obtained the desired section $\widetilde{s} := u - u_1 s_0$, and we have produced infinite dimensionally many L^2 CR sections of $B^{\otimes n}$. □

We define the Kohn Laplacian $\square_M = \overline{\partial}_M \overline{\partial}_M^*$ on $L^{0,1}(M, B^{\otimes n})$ with

$$\mathrm{Dom}(\square_M) := \{v \in L^{0,1}(M, B^{\otimes n}) \mid v \in \mathrm{Dom}(\overline{\partial}_M^*), \overline{\partial}_M^* v \in \mathrm{Dom}(\overline{\partial}_M)\}.$$

From Proposition 3.1, we can see that $\square_M$ has bounded inverse on $L^{0,1}(M, B^{\otimes n})$, and we call it the complex Green operator G_M. The complex Green operator G_M is useful since it induces the canonical solution operator $\overline{\partial}_M^* G_M$ giving the solution $u := \overline{\partial}_M^* G_M v$ to $\overline{\partial}_M u = v$ which has minimum L^2-norm.

In view of Corollary 3.2, to produce CR sections of given transverse regularity, it is enough to find a solution to the tangential Cauchy–Riemann equation having desired regularity. Hence, the problem is reduced to obtain Sobolev estimates of complex Green operator. What Ohsawa and Sibony did in [9] is exactly this.

Theorem 3.3 (cf. [9, Proposition 1] and [5, Proposition 5.3])**.** *For each Sobolev order $k \in \mathbb{N}$, there exists $n = n(k) \in \mathbb{N}$ such that for any $m \geq n$,*

$$\|\overline{\partial}_M^* v\|_{k,m} \leq C_{k,m} \|\square_M v\|_{k,m}$$

holds for any $v \in \Omega^{0,1}(M, B^{\otimes m})$, where $C_{k,m} > 0$ is a constant independent of v, and $\|\cdot\|_{k,m}$ is a transverse L^2-Sobolev norm of order k for the sections of $B^{\otimes m}$.

Corollary 3.4. *For any $k \in \mathbb{N}$ and $m \geq n(k)+1$, the space of H^k CR sections of $B^{\otimes m}$ is infinite dimensional.*

4 Description of L^2 CR sections

In this section, we shall explain Theorem 2.6. Although it is closely related with Corollary 3.4, it seems not easy to obtain the best threshold $n(k)$ for the Sobolev estimates explicitly. Hence, we shall take another approach, describing L^2 CR sections of $B^{\otimes n}$ in certain Fourier series, based on the author's recent study [2].

Sketch of the proof of Theorem 2.6. Let M and $B \to M$ be the example explained in §2. We use a non-holomorphic coordinate (z, τ) of the covering space $\mathbb{D} \times S^1$ of M given by

$$\tau = \frac{t - z}{1 - \overline{z}t}.$$

For each L^2 CR section $s = s(z,t)(dz)^{\otimes n}$ of $B^{\otimes n}$, we expand $s(z,t)$ in τ as $s(z,t) = \sum_{m \in \mathbb{Z}} s_m(z)\tau^m$. Putting

$$\sigma_m := s_m(z)(dz)^{\otimes n} \otimes \left(\frac{\sqrt{2}dz}{1 - |z|^2}\right)^{\otimes m}$$

gives smooth $(n+m)$-differential $\sigma_m \in C^\infty(\Sigma, K_\Sigma^{n+m})$. Since our s is CR, the differentials $\{\sigma_m\}_{m \in \mathbb{Z}}$, which we call the coefficients of s, enjoy a system of the $\overline{\partial}$-equations

$$\overline{\partial}\sigma_m = -\frac{m-1}{\sqrt{2}}\sigma_{m-1} \otimes g, \quad m \in \mathbb{Z} \tag{4.1}$$

where the hyperbolic metric g is identified with $g = 2dz \otimes d\overline{z}/(1 - |z|^2)^2$. Note that this system of the $\overline{\partial}$-equations is decoupled into that for $\{\sigma_m\}_{m \geq 1}$:

$$\overline{\partial}\sigma_1 = 0, \quad \overline{\partial}\sigma_2 = -\frac{1}{\sqrt{2}}\sigma_1 \otimes \omega, \ldots$$

and that for $\{\sigma_m\}_{m \leq 0}$:

$$\overline{\partial}\sigma_0 = \frac{1}{\sqrt{2}}\sigma_{-1} \otimes \omega, \quad \overline{\partial}\sigma_{-1} = \frac{2}{\sqrt{2}}\sigma_{-2} \otimes \omega, \ldots.$$

We shall look at the L^2-norm of the coefficient $\sigma_m \in C^\infty(\Sigma, K_\Sigma^{n+m})$ defined by

$$\|\sigma_m\|_g^2 := \int_\Sigma |s_m(z)|^2 g(z)^{-(n-1)} i dz \wedge d\overline{z}.$$

From the well-known description

$$H^k(S^1) = \{f \in L^2(S^1) \mid \sum_{m \in \mathbb{Z}} (1 + |m|^2)^k |\hat{f}(m)|^2 < \infty\},$$

it follows that our s is H^k if and only if

$$\sum_{m \in \mathbb{Z}} (1 + |m|^2)^k \|\sigma_m\|_g^2 < \infty.$$

Claim 4.1. The space of H^k CR sections of $B^{\otimes n}$ is infinite dimensional if $n \geq k+1$.

Sketch of the proof. We shall construct H^k CR sections of $B^{\otimes n}$ by giving their coefficients $\{\sigma_m\}$ in the following two ways.

1. For given holomorphic $(n+p)$-differential $\sigma_p \in H^0(\Sigma, K_\Sigma^{n+p})$, $p \geq 1$, we inductively define σ_m for $m \geq p+1$ by solving the $\overline{\partial}$-equation (4.1) with the L^2-minimal solution. We let $\sigma_m = 0$ for $m \leq p-1$. Then, using the techniques developed in [2], we can check that $\|\sigma_m\|_g^2 \lesssim m^{-(2n+1)}$ as $m \to \infty$.

2. For given eigenform $\sigma_0 \in C^\infty(\Sigma, K_\Sigma^n)$ of complex Laplacian $\overline{\partial}^*\overline{\partial}$, we inductively define σ_m for $m \leq -1$ so that the $\overline{\partial}$-equation (4.1) holds. We let $\sigma_m = 0$ for $m \geq 1$. Then, again using the techniques developed in [2], we can check that $\|\sigma_m\|_g^2 \lesssim |m|^{-(2n+1)}$ as $m \to -\infty$.

In both cases, the formal CR section $s(z,t) := \sum_{m\in\mathbb{Z}} s_m(z)\tau^m(dz)^{\otimes n}$ of $B^{\otimes n}$ converges in $L^{0,0}(M, B^{\otimes n})$ and yields a H^k CR section of $B^{\otimes n}$ since $n \geq k+1$. Moreover, we can see that bases $\{\sigma_{p,q}\}_{1\leq q\leq \dim H^0(\Sigma, K_\Sigma^{n+p})} \subset H^0(\Sigma, K_\Sigma^{n+p})$ for $p \geq 1$ and a complete system of eigenforms $\{\sigma_{0,r}\}_{r\geq 1} \subset C^\infty(\Sigma, K_\Sigma^n)$ produce linearly independent H^k CR sections of $B^{\otimes n}$. Therefore, the space of H^k CR sections of $B^{\otimes n}$ is infinite dimensional. □

Claim 4.2. The space of H^k CR sections of $B^{\otimes n}$ is finite dimensional if $n \leq k$.

Sketch of the proof. Let s be a H^k CR section of $B^{\otimes n}$. Suppose that there is $j \in \mathbb{N}$ such that $\sigma_j \neq 0$ or $\sigma_{-j} \neq 0$. Then, by comparing the coefficients $\{\sigma_n\}$ of s with the construction in Claim 4.1, we can show that $\|\sigma_m\|_g^2 \gtrsim |m|^{-(2n+1)}$ as $m \to \infty$ or $m \to -\infty$ respectively, and it follows that this s is not H^k since $n \leq k$. Hence, σ_j must vanish when $j \neq 0$ and s must be the pull-back of a holomorphic n-differential σ_0 on Σ. Therefore, the space of H^k CR sections of $B^{\otimes n}$ is finite dimensional. □

These two claims complete the proof of Theorem 2.6. □

We conclude this note with the implication of Theorem 2.6 to Sobolev estimates of complex Green operator.

Corollary 4.3. *The Sobolev estimate in Theorem 3.3 fails if $m \leq k-1$.*

Proof. If the Sobolev estimate holds at $m = k-1$, Corollary 3.4 implies that the space of H^k CR sections of $B^{\otimes k}$ is infinite dimensional. This contradicts with Theorem 2.6. □

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